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ABSTRACT

In this paper, we have studied determinants of bicomplex matrices and investigated their properties. We have introduced bicomplex symmetric matrix, bicomplex skew-symmetric matrix, bicomplex idempotent matrix, bicomplex skew-idempotent matrix, bicomplex involutory matrix, bicomplex skew-involutory matrix, three types of Hermetian and skew-Hermetian matrix, bicomplex orthogonal matrix and three types of unitary matrix and investigated the properties of their determinants.

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1. INTRODUCTION

Throughout this paper, the set of Bicomplex numbers is denoted by C_2 and the sets of complex and real numbers are denoted by C_1 and C_0, respectively. For detail of the theory (cf.**[1]**,**[2]**,**[3]**).

Definition 1.1 (Bicomplex numbers).

The set of Bicomplex Numbers defined as:

$C_2 = \{x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4 : x_1, x_2, x_3, x_4 \in C_0\}$

where , $i_1 \neq i_2$, $i_1^2 = i_2^2 = -1$ and $i_1i_2 = i_2i_1$.

We shall use the notations $C(i_1)$ and $C(i_2)$ for the following sets:

$C(i_1) = \{x + i_1y : x, y \in C_0\}$

$C(i_2) = \{x + i_2y : x, y \in C_0\}$

Definition 1.2 (Hyperbolic numbers). The set of Hyperbolic Numbers is defined as

$H = \{x + i_1i_2y : x, y \in C_0\}$

Definition 1.3 (Idempotent elements). Besides 0 and 1, there are exactly two non – trivial idempotent elements exist in C_2, denoted as e_1 and e_2 and defined as $e_1 = \frac{1+i_1i_2}{2}$ and $e_2 = \frac{1-i_1i_2}{2}$.

Note that $e_1 + e_2 = 1$ and $e_1e_2 = e_2e_1 = 0$.

Definition 1.4 (Cartesian idempotent set). [**4**]

Cartesian idempotent set X determined by X_1 and X_2 is denoted as $X_1 \times_e X_2$ and is defined as

$X = X_1 \times_e X_2$

$= \{\xi \in C_2 : \xi = a\, e_1 + b e_2, (a, b) \in X_1 \times X_2\}$.

(i) The Cartesian idempotent set C_2 determined by $C(i_1)$ is given as :

$C_2 = C(i_1) \times_e C(i_1) = C(i_1)e_1 + C(i_1)e_2$

$= \{\xi \in C_2 : \xi = {}^1\xi\, e_1 + {}^2\xi e_2, ({}^1\xi, {}^2\xi) \in C(i_1) \times C(i_1)\}$

(ii) The Cartesian idempotent set C_2 determined by $C(i_2)$ is given as :

$C_2 = C(i_2) \times_e C(i_2) = C(i_2)e_1 + C(i_2)e_2$

$= \{\xi \in C_2 : \xi = \xi_1\, e_1 + \xi_2 e_2, (\xi_1, \xi_2) \in C(i_2) \times C(i_2)\}$.

1.1 Idempotent Representation of Bicomplex Numbers.

The bicomplex numbers can be represented in two different idempotent forms w.r.t. the elements from $C(i_1)$ and $C(i_2)$, explained as follows :

(a) The $C(i_1)$–idempotent representation of Bicomplex Number is given by

$\xi = x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4 = (x_1 + i_1x_2) + i_2(x_3 + i_1x_4)$

$= z_1 + i_2z_2 = (z_1 - i_1z_2)e_1 + (z_1 + i_1z_2)e_2 = {}^1\xi\, e_1 + {}^2\xi e_2$,

where ${}^1\xi = z_1 - i_1z_2 = (x_1 + x_4) + i_1(x_2 - x_3)$,

${}^2\xi = z_1 + i_1z_2 = (x_1 - x_4) + i_1(x_2 + x_3) \in C(i_1)$

(b) The $C(i_2)$ – idempotent representation of Bicomplex Number is given by

$\xi = x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4 = (x_1 + i_2x_3) + i_1(x_2 + i_2x_4)$

$= w_1 + i_1w_2 = (w_1 - i_2w_2)e_1 + (w_1 + i_2w_2)e_2 = \xi_1 e_1 + \xi_2 e_2,$

where $\xi_1 = w_1 - i_2w_2 = (x_1 + x_4) - i_2(x_2 - x_3)$,

$\xi_2 = w_1 + i_2w_2 = (x_1 - x_4) + i_2(x_2 + x_3) \in C(i_2)$

1.2 Non-Singular and Singular Elements in C_2:

Let $\xi, \eta \in C_2$ s.t. $\xi\eta = \eta\xi = 1$, then η is said to be a multiplicative inverse of ξ. An element which has inverse is said to be Non-Singular (Invertible), and an element which does not have an inverse is said to be Singular (Non-Invertible).

Non-zero singular elements exist in C_2. Set of all singular elements in C_2 is denoted as O_2. C_2/O_2 is the set of all non-singular elements in C_2.

1.2.1 Singular Elements in C_2:

We are providing some conditions for the singularity of the bicomplex number as follows:

(i) $\xi = z_1 + i_2z_2 = (z_1 - i_1z_2)e_1 + (z_1 + i_1z_2)e_2$

$= {}^1\xi e_1 + {}^2\xi e_2$ is singular if and only if

$z_1^2 + z_2^2 = 0$ or $(z_1 - i_1z_2 = 0$ or $z_1 + i_1z_2 = 0)$ or

$({}^1\xi = 0$ or ${}^2\xi = 0)$

(ii) $\xi = w_1 + i_1w_2 = (w_1 - i_2w_2)e_1 + (w_1 + i_2w_2)e_2$

$= \xi_1 e_1 + \xi_2 e_2$ is singular if and only if

$w_1^2 + w_2^2 = 0$ or $(w_1 - i_2w_2 = 0$ or $w_1 + i_2w_2 = 0)$ or

$(\xi_1 = 0$ or $\xi_2 = 0)$

Definition 1.5 (Principal Ideals in C_2).

The principal ideals I_1 and I_2 defined as

$I_1 = \{\xi e_1 : \xi = {}^1\xi e_1 + {}^2\xi e_2 \in C_2\} = \{{}^1\xi e_1 : {}^1\xi \in C(i_1)\}$

$I_2 = \{\xi e_2 : \xi = {}^1\xi e_1 + {}^2\xi e_2 \in C_2\} = \{{}^2\xi e_2 : {}^2\xi \in C(i_1)\}$

Note that $I_1 \cap I_2 = \{0\}$ and $I_1 \cup I_2 = O_2$, set of all singular elements of C_2.

Definition 1.6 (Zero-divisors). Two non-zero bicomplex numbers ξ and η are said to be zero-divisors if $\xi\eta = 0$.

The relation $e_1e_2 = e_2e_1 = 0$ establishes the existence of zero divisors in C_2.

Proposition 1.1. The Bicomplex numbers ξ and η are divisors of zero if and only if $\xi \in I_1/\{0\}$ and $\eta \in I_2/\{0\}$.

Definition 1.7 (Norm on the bicomplex space).

(i) Let $\xi = x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4 = z_1 + i_2z_2$

$= {}^1\xi e_1 + {}^2\xi e_2 \in C_2$

The norm in C_2 is defined as

$$\|\xi\| = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2} = \sqrt{|z_1|^2 + |z_2|^2}$$

$$= \left[\frac{|{}^1\xi|^2 + |{}^2\xi|^2}{2}\right]^{1/2}$$

(ii) Let $\xi = x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4 = w_1 + i_1w_2$

$= \xi_1 e_1 + \xi_2 e_2 \in C_2$

The norm in C_2 is defined as

$$\|\xi\| = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2} = \sqrt{|w_1|^2 + |w_2|^2}$$

$$= \left[\frac{|\xi_1|^2 + |\xi_2|^2}{2}\right]^{1/2}$$

Note that C_2 becomes a modified Banach algebra, in the sense that $\xi, \eta \in C_2$, $\|\xi\eta\| \le \sqrt{2}\|\xi\|\|\eta\|$.

1.3 Conjugations of Bicomplex Numbers.

There are three different types of conjugations given on the bicomplex space and are explained as follows **[3]**:

(a) i_1 − Conjugation

$\xi^* = (z_1 + i_2z_2)^* = \bar{z}_1 + i_2\bar{z}_2, \forall z_1, z_2 \in C(i_1)$

$= ({}^1\xi e_1 + {}^2\xi e_2)^* = \overline{{}^2\xi} e_1 + \overline{{}^1\xi} e_2, \forall {}^1\xi, {}^2\xi \in C(i_1)$

$= (w_1 + i_1w_2)^* = w_1 - i_1w_2, \forall w_1, w_2 \in C(i_2)$

$= (\xi_1 e_1 + \xi_2 e_2)^* = \xi_2 e_1 + \xi_1 e_2, \forall \xi_1, \xi_2 \in C(i_2)$

(b) i_2 − Conjugation

$\xi^\# = (z_1 + i_2z_2)^\# = z_1 - i_2z_2, \forall z_1, z_2 \in C(i_1)$

$= ({}^1\xi e_1 + {}^2\xi e_2)^\# = {}^2\xi e_1 + {}^1\xi e_2, \forall {}^1\xi, {}^2\xi \in C(i_1)$

$= (w_1 + i_1w_2)^\# = \bar{w}_1 + i_1\bar{w}_2, \forall w_1, w_2 \in C(i_2)$

$= (\xi_1 e_1 + \xi_2 e_2)^\# = \overline{\xi_2} e_1 + \overline{\xi_1} e_2, \forall \xi_1, \xi_2 \in C(i_2)$

(c) i_1i_2 − Conjugation

$\xi' = (z_1 + i_2z_2)' = \bar{z}_1 - i_2\bar{z}_2, \forall z_1, z_2 \in C(i_1)$

$= ({}^1\xi e_1 + {}^2\xi e_2)' = \overline{{}^1\xi} e_1 + \overline{{}^2\xi} e_2, \forall {}^1\xi, {}^2\xi \in C(i_1)$

$= (w_1 + i_1w_2)' = \bar{w}_1 - i_1\bar{w}_2, \forall w_1, w_2 \in C(i_2)$

$= (\xi_1 e_1 + \xi_2 e_2)' = \overline{\xi_1} e_1 + \overline{\xi_2} e_2, \forall \xi_1, \xi_2 \in C(i_2)$

2. BICOMPLEX MATRIX

In this section, we discussed about the bicomplex matrices along with their properties and some results. We established the results with respect to the conjugations of the bicomplex matrices. For detail of the theory of Bicomplex matrix cf. **[5, 6, 7, 8, 9]**.

Here, we denote

$C_2^{m\times n} = \left\{A = \left[\xi_{ij}\right]_{m\times n} : \xi_{ij} \in C_2\right\}$ the set of $m \times n$ matrices with bicomplex entries.

Let $A = \left[\xi_{ij}\right]_{m\times n} \in C_2^{m\times n} \Rightarrow \xi_{ij} \in C_2$

2.1 Idempotent Representation of Bicomplex Matrix.

We can represent the Bicomplex Matrix in two different idempotent forms in terms of $C^{m\times n}(i_1)$ and $C^{m\times n}(i_2)$, explained as follows :

(i) The $C^{m\times n}(i_1)$-idempotent representation of Bicomplex Matrix $A = \left[\xi_{ij}\right]_{m\times n} \in C_2^{m\times n}$ is given by

$$A = \left[\xi_{ij}\right]_{m\times n} = \left[{}^1\xi_{ij}\right]_{m\times n} e_1 + \left[{}^2\xi_{ij}\right]_{m\times n} e_2$$
$$= {}^1A\, e_1 + {}^2A\, e_2$$

where ${}^1A = \left[{}^1\xi_{ij}\right]_{m\times n}, {}^2A = \left[{}^2\xi_{ij}\right] \in C^{m\times n}(i_1)$

(ii) The $C^{m\times n}(i_2)$ –idempotent representation of Bicomplex Matrix $A = \left[\xi_{ij}\right]_{m\times n} \in C_2^{m\times n}$ is given by

$$A = \left[\xi_{ij}\right]_{m\times n} = \left[\xi_{1,ij}\right]_{m\times n} e_1 + \left[\xi_{2,ij}\right]_{m\times n} e_2$$
$$= A_1\, e_1 + A_2\, e_2$$

where $A_1 = \left[\xi_{1,ij}\right]_{m\times n}, A_2 = \left[\xi_{2,ij}\right]_{m\times n} \in C^{m\times n}(i_2)$

2.2 Determinant of Bicomplex Matrices. As, only square matrices can have determinant, so let, $A = \left[\xi_{ij}\right]_{n\times n} \in C_2^{n\times n}$. Then determinant of A is denoted as det(A).

Theorem 2.1. Let $A = \left[\xi_{ij}\right]_{n\times n} \in C_2^{n\times n}$, then its determinant is given by

$\det(A) = \det({}^1A)\, e_1 + \det({}^2A)\, e_2$ or

$\det(A) = \det(A_1)\, e_1 + \det(A_2)\, e_2$.

Corollary 2.2. For $A, B \in C_2^{n\times n}$,

$\det(AB) = \det(A)\det(B)$.

2.3 Algebraic Structure of Bicomplex Matrices. Bicomplex Matrices have the following algebraic structures

$C_2^{m\times n}(C_0)$, $C_2^{m\times n}(C(i_1))$ and $C_2^{m\times n}(C(i_2))$ are Linear Space; $C_2^{m\times n}(C_2)$ is a C_2- Module; $C_2^{n\times n}(C_0), C_2^{n\times n}(C(i_1))$ and $C_2^{n\times n}(C(i_2))$ are Algebra.

Theorem 2.3. $A = {}^1A\, e_1 + {}^2A\, e_2 = A_1e_1 + A_2e_2 \in C_2^{n\times n}$ is invertible iff ${}^1A$ and ${}^2A$ are invertible in $C^{n\times n}(i_1)$ and A_1 and A_2 are invertible in $C^{n\times n}(i_2)$.

Corollary 2.4. $A \in C_2^{n\times n}$ is invertible iff det (A) $\notin O_2$.

2.4 Non-Singular and Singular Bicomplex Matrix.

A matrix $A = \left[\xi_{ij}\right]_{n\times n} \in C_2^{n\times n}$ is said to be Non-Singular (Invertible) if $\det(A) \notin O_2$. It is said to be Singular (Non-invertible) matrix if $\det(A) \in O_2$.

2.5. Conjugation of Bicomplex Matrix. Here, we have given three different types of conjugations of the bicomplex matrices. Let $A = \left[\xi_{ij}\right]_{m\times n} \in C_2^{m\times n}$ be a bicomplex matrix. Then, its conjugations are defined as follows:

(a) i_1-Conjugation

$A^* = ({}^1A\, e_1 + {}^2A\, e_2)^* = \overline{{}^2A}\, e_1 + \overline{{}^1A}\, e_2$

when ${}^1A, {}^2A \in C^{m\times n}(i_1)$

$A^* = (A_1\, e_1 + A_2\, e_2)^* = A_2\, e_1 + A_1\, e_2$

when $A_1, A_2 \in C^{m\times n}(i_2)$

(b) i_2- Conjugation

$A^\# = ({}^1A\, e_1 + {}^2A\, e_2)^\# = {}^2A\, e_1 + {}^1A\, e_2$

when ${}^1A, {}^2A \in C^{m\times n}(i_1)$

$A^\# = (A_1\, e_1 + A_2\, e_2)^\# = \overline{A}_2\, e_1 + \overline{A}_1\, e_2$

when $A_1, A_2 \in C^{m\times n}(i_2)$

(c) i_1i_2- Conjugation

$A' = ({}^1A\, e_1 + {}^2A\, e_2)' = \overline{{}^1A}\, e_1 + \overline{{}^2A}\, e_2$

when ${}^1A, {}^2A \in C^{m\times n}(i_1)$

$A' = (A_1\, e_1 + A_2\, e_2)' = \overline{A}_1\, e_1 + \overline{A}_2\, e_2$

when $A_1, A_2 \in C^{m\times n}(i_2)$

2.6 Adjoint of a Bicomplex Square Matrix.

Let $A = {}^1A\, e_1 + {}^2A\, e_2 = A_1e_1 + A_2e_2 \in C_2^{n\times n}$.

Then adjoint of the bicomplex matrix A is given by

$\mathrm{adj}(A) = \mathrm{adj}({}^1A)e_1 + \mathrm{adj}({}^2A)e_2$ or

$\text{adj}(A) = \text{adj}(A_1)e_1 + \text{adj}(A_2)e_2$.

Theorem 2.5. If A is a bicomplex square matrix, then $A\text{adj}(A) = \text{adj}(A)A = \det(A)I$.

Theorem 2.6 (Inverse of a Bicomplex Square Matrix). Let A be a non-singular bicomplex square matrix of order n, then $A^{-1} = \frac{1}{\det(A)}\text{adj}(A)$.

Theorem 2.7. Let A be a non-singular bicomplex square matrix of order n, then

$$A^{-1} = \frac{1}{\det({}^1A)}\text{adj}({}^1A)e_1 + \frac{1}{\det({}^2A)}\text{adj}({}^2A)e_2$$

or $A^{-1} = \frac{1}{\det(A_1)}\text{adj}(A_1)e_1 + \frac{1}{\det(A_2)}\text{adj}(A_2)e_2$.

Theorem 2.8: Let A be a non-singular bicomplex square matrix of order n, then

$$A^{-1} = ({}^1A)^{-1}e_1 + ({}^2A)^{-1}e_2$$

or $A^{-1} = (A_1)^{-1}e_1 + (A_2)^{-1}e_2$.

Theorem 2.9. Let A be a non-singular bicomplex square matrix of order n, then following holds:

(i) A is invertible

(ii) $A^{-1} = \frac{1}{\det(A)}\text{adj}(A)$

(iii) $A^{-1} = \frac{1}{\det({}^1A)}\text{adj}({}^1A)e_1 + \frac{1}{\det({}^2A)}\text{adj}({}^2A)e_2$

(iv) $A^{-1} = \frac{1}{\det(A_1)}\text{adj}(A_1)e_1 + \frac{1}{\det(A_2)}\text{adj}(A_2)e_2$

(v) $A^{-1} = ({}^1A)^{-1}e_1 + ({}^2A)^{-1}e_2$

(vi) $A^{-1} = (A_1)^{-1}e_1 + (A_2)^{-1}e_2$

(vii) $\det(A) \notin O_2$

(viii) $\det({}^1A) \neq 0$ and $\det({}^2A) \neq 0$

(ix) $\det(A_1) \neq 0$ and $\det(A_2) \neq 0$

(x) $\det(A^{-1}) = \frac{1}{\det(A)}$

(xi) $\det(A^{-1}) = \frac{1}{\det({}^1A)} e_1 + \frac{1}{\det({}^2A)} e_2$

(xii) $\det(A^{-1}) = \frac{1}{\det(A_1)} e_1 + \frac{1}{\det(A_2)} e_2$

Corollary 2.10. A matrix $A \in C_2^{n\times n}$, then following conditions are equivalent

(i) A is singular

(ii) $\det(A) \in O_2$

(iii) $\det({}^1A) = 0$ or $\det({}^2A) = 0$

(iv) $\det(A_1) = 0$ or $\det(A_2) = 0$

3. DETERMINANT OF SOME SPECIAL BICOMPLEX MATRICES

In this section, we defined and studied some special type of bicomplex square matrices and their determinant.

Definition 3.1 (Symmetric Matrix). A Bicomplex matrix $A \in C_2^{n\times n}$ is said to be symmetric matrix if $A^t = A$.

Definition 3.2 (Skew-Symmetric Matrix). A Bicomplex matrix $A \in C_2^{n\times n}$ is said to be Skew-Symmetric matrix if $A^t = -A$.

Definition 3.3 (Bicomplex Idempotent Matrix). A Bicomplex matrix $A \in C_2^{n\times n}$ is said to be idempotent matrix if $A^2 = A$.

Definition 3.4 (Bicomplex Skew-Idempotent Matrix). A Bicomplex matrix $A \in C_2^{n\times n}$ is said to be Skew-idempotent matrix if $A^2 = -A$.

Definition 3.5 (Bicomplex Involutory Matrix). A Bicomplex matrix $A \in C_2^{n\times n}$ is said to be involutory matrix if $A^2 = I$.

Definition 3.6 (Bicomplex Skew-Involutory Matrix). A Bicomplex matrix $A \in C_2^{n\times n}$ is said to be Skew-involutory matrix if $A^2 = -I$.

Definition 3.7 (Bicomplex Hermitian Matrix). There are three types of Bicomplex Hermitian Matrix, defined as follows:

(i) $i_1 -$ Hermitian Matrix

A Bicomplex matrix $A \in C_2^{n\times n}$ is said to be $i_1 -$ Hermitian Matrix if $(A^t)^* = (A^*)^t = A$.

(ii) $i_2 -$ Hermitian Matrix

A Bicomplex matrix $A \in C_2^{n\times n}$ is said to be $i_2 -$ Hermitian Matrix if $(A^t)^\# = \left(A^\#\right)^t = A$.

(iii) $i_1\, i_2 -$ Hermitian Matrix

A Bicomplex matrix $A \in C_2^{n\times n}$ is said to be $i_1\, i_2 -$ Hermitian Matrix if $(A^t)' = (A')^t = A$.

Note 3.1. There are some observations as given follows:

(a) If A is $i_1 -$ Hermitian Matrix then $A^* = A^t$.

(b) If A is $i_2 -$ Hermitian Matrix then $A^\# = A^t$

(c) If A is $i_1 i_2 -$ Hermitian Matrix then $A' = A^t$.

Definition 3.8 (Bicomplex Skew- Hermitian Matrix). There are three types of bicomplex skewHermitian matrix, defined as follows:

(i) $i_1 - \text{skew} -$ Hermitian Matrix

A Bicomplex matrix $A \in C_2^{n\times n}$ is said to be $i_1 - \text{skew} -$ Hermitian Matrix if

$$(A^t)^* = (A^*)^t = -A\,.$$

(ii) $i_2 - \text{skew} -$ Hermitian Matrix

A Bicomplex matrix $A \in C_2^{n\times n}$ is said to be $i_2 - \text{skew} -$ Hermitian Matrix if

$$(A^t)^\# = (A^\#)^t = -A\,.$$

(iii) $i_1\, i_2 - \text{skew} -$ Hermitian Matrix

A Bicomplex matrix $A \in C_2^{n\times n}$ is said to be $i_1\, i_2 - \text{skew} -$ Hermitian Matrix if

$$(A^t)' = (A')^t = -A\,.$$

Note 3.2. Some observations about the definitions are given as follows:

(a) If A is $i_1 - \text{skew} -$ Hermitian Matrix then $A^* = -A^t$.

(b) If A is $i_2 - \text{skew} -$ Hermitian Matrix then $A^\# = -A^t$.

(c) If A is $i_1\, i_2 - \text{skew} -$ Hermitian Matrix then $A' = -A^t$.

Definition 3.9 (Bicomplex Orthogonal Matrix). A Bicomplex matrix $A \in C_2^{n\times n}$ is said to be orthogonal matrix if $A^tA = AA^t = I$.

Definition 3.10 (Bicomplex Unitary Matrix).
Unitary Matrix, defined as follows:

(i) $i_1 -$ Unitary Matrix

A Bicomplex matrix $A \in C_2^{n\times n}$ is said to be $i_1 -$ Unitary Matrix if $A^{*t}A = A^{*t}A = I$.

(ii) $i_2 -$ Unitary Matrix

A Bicomplex matrix $A \in C_2^{n\times n}$ is said to be $i_2 -$ Unitary Matrix if $A^{\#t}A = A^{\#t}A = I$.

(iii) $i_1\, i_2 -$ Unitary Matrix

A Bicomplex matrix $A \in C_2^{n\times n}$ is said to be $i_1\, i_2 -$ Unitary Matrix if $A'^tA = A'^tA = I$.

Proposition 3.1: Let $A \in C_2^{n\times n}$ be a Bicomplex Skew-symmetric matrix of odd order. Then $\det(A) = 0$.

Proof: Let $A \in C_2^{n\times n}$ be a Skew-symmetric matrix of odd order

$\Rightarrow A^t = -A$

$\Rightarrow \det(A^t) = \det(-A)$

$\Rightarrow \det(A) = (-1)^n \det(A)$

$\Rightarrow \det(A) = -\det(A) \qquad (\because (-1)^n = -1, \text{n is odd})$

$\Rightarrow 2\det(A) = 0$

$\Rightarrow \det(A) = 0$

Proposition 3.2: Let $A \in C_2^{n\times n}$ be a Bicomplex idempotent matrix. Then $\det(A) \in \{0,1,e_1,e_2\}$.

Proof: Let $A \in C_2^{n\times n}$ be an idempotent matrix

$\Rightarrow A^2 = A$

$\Rightarrow \det(A^2) = \det(A)$

$\Rightarrow \det(AA) = \det(A)$

$\Rightarrow \det(A)\det(A) = \det(A)$

$\Rightarrow \{\det(A)\}^2 = \det(A)$

$\Rightarrow \{\det(^1A)\, e_1 + \det(^2A)\, e_2\}^2 = \det(^1A)\, e_1 + \det(^2A)\, e_2$

$\Rightarrow \{\det(^1A)\}^2 e_1 + \{\det(^2A)\}^2 e_2 = \det(^1A)\, e_1 + \det(^2A)\, e_2$

$\Rightarrow \{\det(^1A)\}^2 = \det(^1A)$ and $\{\det(^2A)\}^2 = \det(^2A)$

$\Rightarrow \det(^1A) = 0,1$ and $\det(^2A) = 0,1$

Hence $\det(A) \in \{0,1,e_1,e_2\}$.

Note 3.3: If A is a non-singular Bicomplex idempotent matrix then $A = I$.

Proposition 3.3: Let $A \in C_2^{n\times n}$ be a Bicomplex skew-idempotent matrix. Then $\det(A) \in \{0,1,e_1,e_2\}$ when n is even and $\det(A) \in \{0,-1,-e_1,-e_2\}$ when n is odd.

Proof: Let $A \in C_2^{n\times n}$ be a Bicomplex skew-idempotent matrix

$\Rightarrow A^2 = -A$

$\Rightarrow \det(A^2) = \det(-A)$

$\Rightarrow \det(AA) = (-1)^n \det(A)$

$\Rightarrow \det(A)\det(A) = (-1)^n \det(A)$

$\Rightarrow \{\det(A)\}^2 = (-1)^n \det(A)$

Case(i) When n is even

$\Rightarrow \{\det(A)\}^2 = \det(A)\ (\because (-1)^n = 1, \text{n is even})$

The result directly follows **Proposition 3.2**

Case(ii) When n is odd

$\Rightarrow \{\det(A)\}^2 = -\det(A)\ (\because (-1)^n = -1, n \text{ is odd})$

$\Rightarrow \{\det({}^1A)e_1 + \det({}^2A)e_2\}^2 = -\{\det({}^1A)e_1 + \det({}^2A)e_2\}$

$\Rightarrow \{\det({}^1A)\}^2 e_1 + \{\det({}^2A)\}^2 e_2 = -\det({}^1A)\, e_1 - \det({}^2A)\, e_2$

$\Rightarrow \{\det({}^1A)\}^2 = -\det({}^1A)$ and $\{\det({}^2A)\}^2 = -\det({}^2A)$

$\Rightarrow \det({}^1A) = 0, -1$ and $\det({}^2A) = 0, -1$

Hence $\det(A) \in \{0, -1, -e_1, -e_2\}$.

Note 3.4: If A is a non-singular Bicomplex Skew-idempotent matrix then $A = -I$.

Proposition 3.4. Let $A \in C_2^{n\times n}$ such that $A^2 = \eta A$, $\eta \in C_2$. Then $\det(A) \in \{0, (\eta)^n, ({}^1\eta)^n e_1, ({}^2\eta)^n e_2\}$.

Proof: Let $A \in C_2^{n\times n}$ such that $A^2 = \eta A, \eta \in C_2$

$\Rightarrow A^2 = \eta A$

$\Rightarrow \det(A^2) = \det(\eta A)$

$\Rightarrow \det(AA) = (\eta)^n \det(A)$

$\Rightarrow \det(A)\det(A) = (\eta)^n \det(A)$

$\Rightarrow \{\det(A)\}^2 = (\eta)^n \det(A)$

$\Rightarrow \{\det({}^1A)\, e_1 + \det({}^2A)\, e_2\}^2$

$= \{({}^1\eta)^n\, e_1 + ({}^2\eta)^n\, e_2\}\{\det({}^1A)e_1 + \det({}^2A)e_2\}$

$\Rightarrow \{\det({}^1A)\}^2 e_1 + \{\det({}^2A)\}^2 e_2 = ({}^1\eta)^n \det({}^1A)\, e_1 + ({}^2\eta)^n \det({}^2A)\, e_2$

$\Rightarrow \{\det({}^1A)\}^2 = ({}^1\eta)^n \det({}^1A)$ and $\{\det({}^2A)\}^2 = ({}^2\eta)^n \det({}^2A)$

$\Rightarrow \det({}^1A) = 0, ({}^1\eta)^n$ and $\det({}^2A) = 0, ({}^2\eta)^n$

Hence $\det(A) \in \{0, (\eta)^n, ({}^1\eta)^n e_1, ({}^2\eta)^n e_2\}$.

Note 3.5. (i) For $\eta = 1$, we get **Proposition 3.2**

(ii) For $\eta = -1$, we get **Proposition 3.3**

(iii) If A is non-singular, then $A = \eta I$

Proposition 3.5: Let $A \in C_2^{n\times n}$ be a Bicomplex involutory matrix. Then $\det(A) \in \{1, -1, i_1 i_2, -i_1 i_2\}$.

Proof: Let A be any Bicomplex involutory matrix

$\Rightarrow A^2 = I$

$\Rightarrow \det(A^2) = \det(I)$

$\Rightarrow \det(AA) = 1$

$\Rightarrow \det(A)\det(A) = 1$

$\Rightarrow \{\det(A)\}^2 = 1$

$\Rightarrow \{\det({}^1A)e_1 + \det({}^2A)e_2\}^2 = 1e_1 + 1e_2$

$\Rightarrow \{\det({}^1A)\}^2 e_1 + \{\det({}^2A)\}^2 e_2 = 1e_1 + 1e_2$

$\Rightarrow \{\det({}^1A)\}^2 = 1$ and $\{\det({}^2A)\}^2 = 1$

$\Rightarrow \det({}^1A) = 1, -1$ and $\det({}^2A) = 1, -1$

Hence $\det(A) \in \{1, -1, i_1 i_2, -i_1 i_2\}$.

Note 3.6. Bicomplex involutory matrices are always non-singular and $A^{-1} = A$.

Proposition 3.6. Let $A \in C_2^{n\times n}$ be a Bicomplex Skew-involutory matrix. Then $\det(A) \in \{1, -1, i_1 i_2, -i_1 i_2\}$ when n is even and $\det(A) \in \{i_1, -i_1, i_2, -i_2\}$ when n is odd.

Proof: Let $A \in C_2^{n\times n}$ be a Bicomplex Skew-involutory matrix

$\Rightarrow A^2 = -I$

$\Rightarrow \det(A^2) = \det(-I)$

$\Rightarrow \det(AA) = (-1)^n$

$\Rightarrow \det(A)\det(A) = (-1)^n$

$\Rightarrow \{\det(A)\}^2 = (-1)^n$

Case(i) When n is even

$\Rightarrow \{\det(A)\}^2 = 1 \quad (\because (-1)^n = 1, n \text{ is even})$

The result directly follows **Proposition 3.5**

Case(ii) When n is odd

$\Rightarrow \{\det(A)\}^2 = -1 \quad (\because (-1)^n = -1, n \text{ is odd})$

$\Rightarrow \{\det({}^1A)e_1 + \det({}^2A)e_2\}^2 = -1e_1 - 1e_2$

$\Rightarrow \{\det({}^1A)\}^2 e_1 + \{\det({}^2A)\}^2 e_2 = -1e_1 - 1e_2$

$\Rightarrow \{\det({}^1A)\}^2 = -1$ and $\{\det({}^2A)\}^2 = -1$

$\Rightarrow \det({}^1A) = i_1, -i_1$ and $\det({}^2A) = i_1, -i_1$

Hence $\det(A) \in \{i_1, -i_1, i_2, -i_2\}$.

Note 3.7. Bicomplex skew-involutory matrices are always non-singular and $A^{-1} = -A$.

Proposition 3.7. Let $A \in C_2^{n\times n}$ such that $A^2 = \eta I, \eta \in C_2$. Then,

$$\det(A) \in \left\{\begin{matrix} \sqrt{({}^1\eta)^n}\, e_1 + \sqrt{({}^2\eta)^n}\, e_2, -\sqrt{({}^1\eta)^n}\, e_1 - \sqrt{({}^2\eta)^n}\, e_2, \\ \sqrt{({}^1\eta)^n}\, e_1 - \sqrt{({}^2\eta)^n}\, e_2, -\sqrt{({}^1\eta)^n}\, e_1 + \sqrt{({}^2\eta)^n}\, e_2 \end{matrix}\right\}.$$

Proof: Let $A \in C_2^{n\times n}$ be any Bicomplex matrix such that $A^2 = \eta I$

$\Rightarrow A^2 = \eta I$

$\Rightarrow \det(A^2) = \det(\eta I)$

$\Rightarrow \det(AA) = \eta^n \det(I)$

$\Rightarrow \det(A)\det(A) = \eta^n$

$\Rightarrow \{\det(A)\}^2 = \eta^n$

$\Rightarrow \{\det({}^1A)e_1 + \det({}^2A)e_2\}^2 = ({}^1\eta)^n\, e_1 + ({}^2\eta)^n\, e_2$

$\Rightarrow \{\det({}^1A)\}^2 e_1 + \{\det({}^2A)\}^2 e_2$

$= ({}^1\eta)^n\, e_1 + ({}^2\eta)^n\, e_2$

$\Rightarrow \{\det({}^1A)\}^2 = ({}^1\eta)^n$ and $\{\det({}^2A)\}^2 = ({}^2\eta)^n$

$\Rightarrow \det({}^1A) = \sqrt{({}^1\eta)^n}, -\sqrt{({}^1\eta)^n}$ and

$\det({}^2A) = \sqrt{({}^2\eta)^n}, -\sqrt{({}^2\eta)^n}$

Hence

$$\det(A) \in \left\{ \begin{matrix} \sqrt{({}^1\eta)^n}\, e_1 + \sqrt{({}^2\eta)^n}\, e_2, -\sqrt{({}^1\eta)^n}\, e_1 - \sqrt{({}^2\eta)^n}\, e_2, \\ \sqrt{({}^1\eta)^n}\, e_1 - \sqrt{({}^2\eta)^n}\, e_2, -\sqrt{({}^1\eta)^n}\, e_1 + \sqrt{({}^2\eta)^n}\, e_2 \end{matrix} \right\}.$$

Note 3.8. (i) For $\eta = 1$, the **Proposition 3.5** holds.

(ii) For $\eta = -1$, the **Proposition 3.6** holds.

Proposition 3.8: Let $A \in C_2^{n\times n}$ be a i_1-Hermitian matrix. Then

(i) $\overline{\det({}^2A)} = \det({}^1A)$ or equivalently $\overline{\det({}^1A)} = \det({}^2A)$

(ii) $\det(A) = \overline{\det({}^2A)}\, e_1 + \overline{\det({}^1A)}\, e_2$

(iii) $\det(A_1) = \det(A_2)$

(iv) $\det(A) = \det(A_1) = \det(A_2)$

(v) $\det(A) \in C(i_2)$

Proof: (i) Let $A \in C_2^{n\times n}$ be a i_1-Hermitian matrix

$\Rightarrow A^* = A^t$

$\Rightarrow \det(A^*) = \det(A^t)$

$\Rightarrow \det(A^*) = \det(A) \qquad (\because \det(A^t) = \det(A))$

$\Rightarrow [\det(A)]^* = \det(A) \qquad (\because \det(A^*) = [\det(A)]^*)$

$\Rightarrow [\det({}^1A)e_1 + \det({}^2A)e_2]^* = \det({}^1A)e_1 + \det({}^2A)e_2$

$\Rightarrow \overline{\det({}^1A)}e_2 + \overline{\det({}^2A)}e_1 = \det({}^1A)e_1 + \det({}^2A)\, e_2$

$\Rightarrow \overline{\det({}^2A)}e_1 + \overline{\det({}^1A)}e_2 = \det({}^1A)e_1 + \det({}^2A)\, e_2$

$\Rightarrow \overline{\det({}^2A)} = \det({}^1A)$ and $\overline{\det({}^1A)} = \det({}^2A)$

$\Rightarrow \overline{\det({}^2A)} = \det({}^1A)$ or equivalently $\overline{\det({}^1A)} = \det({}^2A)$

(ii) From **(i)**

$\overline{\det({}^2A)} = \det({}^1A)$ or equivalently $\overline{\det({}^1A)} = \det({}^2A)$

$\Rightarrow \det({}^1A) = \overline{\det({}^2A)}$ and $\det({}^2A) = \overline{\det({}^1A)}$

$\Rightarrow \det({}^1A)e_1 + \det({}^2A)e_2 = \overline{\det({}^2A)}\, e_1 + \overline{\det({}^1A)}\, e_2$

$\Rightarrow \det(A) = \overline{\det({}^2A)}\, e_1 + \overline{\det({}^1A)}\, e_2$

(iii) Let $A \in C_2^{n\times n}$ be a i_1-Hermitian matrix

$\Rightarrow A^* = A^t$

$\Rightarrow \det(A^*) = \det(A^t)$

$\Rightarrow \det(A^*) = \det(A) \qquad (\because \det(A^t) = \det(A))$

$\Rightarrow [\det(A)]^* = \det(A) \qquad (\because \det(A^*) = [\det(A)]^*)$

$\Rightarrow [\det(A_1)e_1 + \det(A_2)e_2]^* = \det(A_1)e_1 + \det(A_2)e_2$

$\Rightarrow \det(A_1)e_2 + \det(A_1)e_1 = \det(A_1)e_1 + \det(A_2)e_2$

$\Rightarrow \det(A_2)e_1 + \det(A_1)e_2 = \det(A_1)e_1 + \det(A_2)e_2$

$\Rightarrow \det(A_2) = \det(A_1)$ and $\det(A_1) = \det(A_2)$

$\Rightarrow \det(A_1) = \det(A_2)$

(iv) From **(iii)**

$\det(A_1) = \det(A_2)$

$\Rightarrow \det(A) = \det(A_1) = \det(A_2)$

(v) From **(iv)** $\det(A) = \det(A_1) = \det(A_2)$

Also, $\det(A_1) = \det(A_2) \in C(i_2)$

$\Rightarrow \det(A) \in C(i_2)$

Proposition 3.9. Let $A \in C_2^{n\times n}$ be a i_2-Hermitian matrix. Then

(i) $\det({}^1A) = \det({}^2A)$

(ii) $\det(A) = \det({}^1A) = \det({}^2A)$

(iii) $\overline{\det(A_2)} = \det(A_1)$ or equivalently $\overline{\det(A_1)} = \det(A_2)$

(iv) $\det(A) = \overline{\det(A_2)}\, e_1 + \overline{\det(A_1)}e_2$

(v) $\det(A) \in C(i_1)$

Proof:

(i) Let $A \in C_2^{n\times n}$ be a i_2-Hermitian matrix

$\Rightarrow A^{\#} = A^t$

$\Rightarrow \det(A^{\#}) = \det(A^t)$

$\Rightarrow \det(A^{\#}) = \det(A) \qquad (\because \det(A^t) = \det(A))$

$\Rightarrow [\det(A)]^{\#} = \det(A) \qquad (\because \det(A^{\#}) = [\det(A)]^{\#})$

$\Rightarrow [\det({}^1A)e_1 + \det({}^2A)\, e_2]^{\#} = \det({}^1A)e_1 + \det({}^2A)e_2$

$\Rightarrow \det({}^1A)e_2 + \det({}^2A)\, e_1 = \det({}^1A)e_1 + \det({}^2A)e_2$

$\Rightarrow \det({}^2A)e_1 + \det({}^1A)e_2 = \det({}^1A)e_1 + \det({}^2A)e_2$

$\Rightarrow \det({}^2A) = \det({}^1A)$ and $\det({}^1A) = \det({}^2A)$

$\Rightarrow \det({}^1A) = \det({}^2A)$

(ii) From **(i)** $\det({}^1A) = \det({}^2A)$

$\Rightarrow \det(A) = \det({}^1A) = \det({}^2A)$

(iii) Let $A \in C_2^{n\times n}$ be a i_2-Hermitian matrix

$\Rightarrow A^{\#} = A^t$

$\Rightarrow \det(A^{\#}) = \det(A^t)$

$\Rightarrow \det(A^{\#}) = \det(A) \quad (\because \det(A^t) = \det(A))$

$\Rightarrow [\det(A)]^{\#} = \det(A) \quad (\because \det(A^{\#}) = [\det(A)]^{\#})$

$\Rightarrow [\det(A_1)e_1 + \det(A_2)e_2]^{\#}$

$= \det(A_1)e_1 + \det(A_2)e_2$

$\Rightarrow \overline{\det(A_1)}e_2 + \overline{\det(A_2)}e_1 = \det(A_1)e_1 + \det(A_2)e_2$

$\Rightarrow \overline{\det(A_2)}e_1 + \overline{\det(A_1)}e_2 = \det(A_1)e_1 + \det(A_2)e_2$

$\Rightarrow \overline{\det(A_2)} = \det(A_1)$ and $\overline{\det(A_1)} = \det(A_2)$

$\Rightarrow \overline{\det(A_2)} = \det(A_1)$ or equivalently

$\overline{\det(A_1)} = \det(A_2)$

(iv) From **(iii)**

$\overline{\det(A_2)} = \det(A_1)$ or equivalently

$\overline{\det(A_1)} = \det(A_2)$

$\Rightarrow \det(A) = \overline{\det(A_2)}\, e_1 + \overline{\det(A_1)} e_2$

(v) From **(ii)**

$\det(A) = \det({}^1A) = \det({}^2A)$

Also $\det({}^1A) = \det({}^2A) \in C(i_1)$

$\Rightarrow \det(A) \in C(i_1)$

Proposition 3.10: Let $A \in C_2^{n\times n}$ be a $i_1 i_2$ −Hermitian matrix. Then

(i) $\det({}^1A)$ and $\det({}^2A)$ both are real

(ii) $\det(A) = \overline{\det({}^1A)}\, e_1 + \overline{\det({}^2A)}\, e_2$

(iii) $\det(A_1)$ and $\det(A_2)$ both are real

(iv) $\det(A) = \overline{\det(A_1)}\, e_1 + \overline{\det(A_2)} e_2$

(v) $\det(A) \in H$

Proof:

(i) Let $A \in C_2^{n\times n}$ be a $i_1 i_2$ −Hermitian matrix

$\Rightarrow A' = A^t$

$\Rightarrow \det(A') = \det(A^t)$

$\Rightarrow \det(A') = \det(A) \quad (\because \det(A^t) = \det(A))$

$\Rightarrow [\det(A)]' = \det(A) \quad (\because \det(A') = [\det(A)]')$

$\Rightarrow [\det({}^1A)e_1 + \det({}^2A)\, e_2]'$

$= \det({}^1A)e_1 + \det({}^2A)e_2$

$\Rightarrow \overline{\det({}^1A)}e_1 + \overline{\det({}^2A)}e_2$

$= \det({}^1A)e_1 + \det({}^2A)e_2$

$\Rightarrow \overline{\det({}^1A)} = \det({}^1A)$ and $\overline{\det({}^2A)} = \det({}^2A)$

$\Rightarrow \det({}^1A)$ is a real and $\det({}^2A)$ is a real

(ii) From **(i)**

$\det({}^1A)$ and $\det({}^2A)$ both are real

$\Rightarrow \det({}^1A) = \overline{\det({}^1A)}$and $\det({}^2A) = \overline{\det({}^2A)}$

$\Rightarrow \det({}^1A)e_1 + \det({}^2A)e_2$

$= \overline{\det({}^1A)}\, e_1 + \overline{\det({}^2A)}\, e_2$

$\Rightarrow \det(A) = \overline{\det({}^1A)}\, e_1 + \overline{\det({}^2A)}\, e_2$

(iii) Let $A \in C_2^{n\times n}$ be a $i_1 i_2$ −Hermitian matrix

$\Rightarrow A' = A^t$

$\Rightarrow \det(A') = \det(A^t)$

$\Rightarrow \det(A') = \det(A) \quad (\because \det(A^t) = \det(A))$

$\Rightarrow [\det(A)]' = \det(A) \quad (\because \det(A') = [\det(A)]')$

$\Rightarrow [\det(A_1)e_1 + \det(A_2)e_2]'$

$= \det(A_1)e_1 + \det(A_2)e_2$

$\Rightarrow \overline{\det(A_1)}e_1 + \overline{\det(A_2)}e_2 = \det(A_1)e_1 + \det(A_2)e_2$

$\Rightarrow \overline{\det(A_1)} = \det(A_1)$ and $\overline{\det(A_2)} = \det(A_2)$

$\Rightarrow \det(A_1)$ is real and $\det(A_2)$ is real

(iv) From **(iii)**

$\det(A_1)$ and $\det(A_2)$ both are real

$\Rightarrow \det(A_1) = \overline{\det(A_1)}$and $\det(A_2) = \overline{\det(A_2)}$

$\Rightarrow \det(A_1)e_1 + \det(A_2)e_2 = \overline{\det(A_1)}e_1 + \overline{\det(A_2)}e_2$

$\Rightarrow \det(A) = \overline{\det(A_1)}e_1 + \overline{\det(A_2)}e_2$

(v) From **(i)**

$\det({}^1A)$ and $\det({}^2A)$ both are real

$\Rightarrow \det(A) \in H$

Proposition 3.11: Let $A \in C_2^{n\times n}$ be a $i_1 -$ Skew − Hermitian matrix. Then the following conditions hold:

(a) When n is even

(i) $\overline{\det({}^2A)} = \det({}^1A)$ or equivalently $\overline{\det({}^1A)} = \det({}^2A)$

(ii) $\det(A) = \overline{\det({}^2A)}\, e_1 + \overline{\det({}^1A)}\, e_2$

(iii) $\det(A_1) = \det(A_2)$

(iv) $\det(A) = \det(A_1) = \det(A_2)$

(v) $\det(A) \in C(i_2)$

(b) When n is odd

(i) $\overline{\det({}^2A)} = -\det({}^1A)$ or equivalently $\overline{\det({}^1A)} = -\det({}^2A)$

(ii) $\det(A) = \overline{-\det({}^2A)}\, e_1 - \overline{\det({}^1A)}\, e_2$

(iii) $\det(A_1) = -\det(A_2)$

(iv) $\det(A) = i_1 i_2 \det(A_1) = -i_1 i_2 \det(A_2)$

(v) $\det(A) \in i_1 C(i_2)$

Proof: Let $A \in C_2^{n\times n}$ be a $i_1 -$ Skew − Hermitian matrix

$\Rightarrow A^* = -A^t$

$\Rightarrow \det(A^*) = \det(-A^t)$

$\Rightarrow \det(A^*) = (-1)^n \det(A^t)$

$\Rightarrow \det(A^*) = (-1)^n \det(A) \quad (\because \det(A^t) = \det(A))$

$\Rightarrow [\det(A)]^* = (-1)^n \det(A) (\because \det(A^*) = [\det(A)]^*)$

(a) When n is even

$[\det(A)]^* = \det(A) \quad (\because (-1)^n = 1, n \text{ is even})$

All the results directly follow **Proposition 3.8**

(b) When n is odd

$[\det(A)]^* = -\det(A) \quad (\because (-1)^n = 1, n \text{ is odd})$

(i) $[\det(A)]^* = -\det(A)$

$\Rightarrow [\det({}^1A)e_1 + \det({}^2A)e_2]^* = -\det({}^1A)e_1 - \det({}^2A)e_2$

$\Rightarrow \overline{\det({}^1A)}e_2 + \overline{\det({}^2A)}e_1 = -\det({}^1A)e_1 - \det({}^2A)e_2$

$\Rightarrow \overline{\det({}^2A)}e_1 + \overline{\det({}^1A)}e_2 = -\det({}^1A)e_1 - \det({}^2A)e_2$

$\Rightarrow \overline{\det({}^2A)} = -\det({}^1A)$ and $\overline{\det({}^1A)} = -\det({}^2A)$

$\Rightarrow \overline{\det({}^2A)} = -\det({}^1A)$ or equivalently

$\overline{\det({}^1A)} = -\det({}^2A)$

(ii) From **(i)** $\overline{\det({}^2A)} = -\det({}^1A)$ or equivalently

$\overline{\det({}^1A)} = -\det({}^2A)$

$\Rightarrow \det(A) = \overline{-\det({}^2A)}\, e_1 - \overline{\det({}^1A)}\, e_2$

(iii) $[\det(A)]^* = -\det(A)$

$\Rightarrow [\det(A_1)e_1 + \det(A_2)e_2]^* = -\det(A_1)e_1 - \det(A_2)e_2$

$\Rightarrow \det(A_1)e_2 + \det(A_1)e_1 = -\det(A_1)e_1 - \det(A_2)e_2$

$\Rightarrow \det(A_2)e_1 + \det(A_1)e_2 = -\det(A_1)e_1 - \det(A_2)e_2$

$\Rightarrow \det(A_2) = -\det(A_1)$ and $\det(A_1) = -\det(A_2)$

$\Rightarrow \det(A_1) = -\det(A_2)$

(iv) From **(iii)** $\det(A_1) = -\det(A_2)$

$\Rightarrow \det(A) = i_1 i_2 \det(A_1) = -i_1 i_2 \det(A_2)$

(v) From **(iv)**

$\det(A) = i_1 i_2 \det(A_1) = -i_1 i_2 \det(A_2)$

Also, $\det(A_1) \in C(i_2)$

$\Rightarrow i_2 \det(A_1) \in C(i_2)$

$\Rightarrow i_1 i_2 \det(A_1) \in i_1 C(i_2)$

$\Rightarrow \det(A) \in i_1 C(i_2)$

Proposition 3.12: Let $A \in C_2^{n\times n}$ be a $i_2 -$ Skew $-$Hermitian matrix. Then following results hold:

(a) When n is even

(i) $\det({}^1A) = \det({}^2A)$

(ii) $\det(A) = \det({}^1A) = \det({}^2A)$

(iii) $\overline{\det(A_2)} = \det(A_1)$ or equivalently $\overline{\det(A_1)} = \det(A_2)$

(iv) $\det(A) = \overline{\det(A_2)}\, e_1 + \overline{\det(A_1)} e_2$

(v) $\det(A) \in C(i_1)$

(b) When n is odd

(i) $\det({}^1A) = -\det({}^2A)$

(ii) $\det(A) = i_1 i_2 \det({}^1A) = -i_1 i_2 \det({}^2A)$

(iii) $\overline{\det(A_2)} = -\det(A_1)$ or equivalently $\overline{\det(A_1)} = -\det(A_2)$

(iv) $\det(A) = -\overline{\det(A_2)}\, e_1 - \overline{\det(A_1)} e_2$

(v) $\det(A) \in i_2 C(i_1)$

Proof:

Let $A \in C_2^{n\times n}$ be a $i_2 -$ Skew $-$Hermitian matrix

$\Rightarrow A^\# = -A^t$

$\Rightarrow \det(A^\#) = \det(-A^t)$

$\Rightarrow \det(A^\#) = (-1)^n \det(A^t)$

$\Rightarrow \det(A^\#) = (-1)^n \det(A) \quad (\because \det(A^t) = \det(A))$

$\Rightarrow [\det(A)]^\# = (-1)^n \det(A) \quad (\because \det(A^\#) = [\det(A)]^\#)$

(a) When n is even

$[\det(A)]^\# = \det(A) \quad (\because (-1)^n = 1, n \text{ is even})$

All the results directly follow **Proposition 3.9**

(b) When n is odd

$[\det(A)]^\# = -\det(A) \quad (\because (-1)^n = -1, n \text{ is odd})$

(i) $[\det(A)]^\# = -\det(A)$

$\Rightarrow [\det({}^1A)e_1 + \det({}^2A)e_2]^\# = -\det({}^1A)e_1 - \det({}^2A)e_2$

$\Rightarrow \det({}^1A)e_2 + \det({}^2A)e_1 = -\det({}^1A)e_1 - \det({}^2A)e_2$

$\Rightarrow \det({}^2A)e_1 + \det({}^1A)e_2 = -\det({}^1A)e_1 - \det({}^2A)e_2$

$\Rightarrow \det({}^2A) = -\det({}^1A)$ and $\det({}^1A) = -\det({}^2A)$

$\Rightarrow \det({}^1A) = -\det({}^2A)$

(ii) From **(i)** $\det({}^1A) = -\det({}^2A)$

$\Rightarrow \det(A) = i_1 i_2 \det({}^1A) = -i_1 i_2 \det({}^2A)$

(iii) $[\det(A)]^\# = -\det(A)$

$\Rightarrow [\det(A_1)e_1 + \det(A_2)e_2]^\# = -\det(A_1)e_1 - \det(A_2)e_2$

$\Rightarrow \overline{\det(A_1)}e_2 + \overline{\det(A_2)}e_1 = -\det(A_1)e_1 - \det(A_2)e_2$

$\Rightarrow \overline{\det(A_2)}\ e_1 + \overline{\det(A_1)}\ e_2$

$= -\det(A_1)e_1 - \det(A_2)e_2$

$\Rightarrow \overline{\det(A_2)} = -\det(A_1)$ and $\overline{\det(A_1)} = -\det(A_2)$

$\Rightarrow \overline{\det(A_2)} = -\det(A_1)$ or equivalently

$\overline{\det(A_1)} = -\det(A_2)$

(iv) From **(iii)**

$\overline{\det(A_2)} = -\det(A_1)$ or equivalently

$\overline{\det(A_1)} = -\det(A_2)$

$\Rightarrow \det(A) = -\overline{\det(A_2)}\ e_1 - \overline{\det(A_1)}e_2$

(v) From **(ii)**

$\det(A) = i_1 i_2 \det({}^1A) = -i_1 i_2 \det({}^2A)$

Also,

$\det({}^1A) \in C(i_1)$

$\Rightarrow i_1 \det({}^1A) \in C(i_1)$

$\Rightarrow i_1 i_2 \det({}^1A) \in i_2 C(i_1)$

$\Rightarrow \det(A) \in i_2 C(i_1)$

Proposition 3.13: Let $A \in C_2^{n\times n}$ be a $i_1 i_2 -$ Skew $-$Hermitian matrix. Then the following results hold:

(a) When n is even

(i) $\det({}^1A)$ and $\det({}^2A)$ both are real

(ii) $\det(A) = \overline{\det({}^1A)}\ e_1 + \overline{\det({}^2A)}\ e_2$

(iii) $\det(A_1)$ and $\det(A_2)$ both are real

(iv) $\det(A) = \overline{\det(A_1)}\ e_1 + \overline{\det(A_2)}e_2$

(v) $\det(A) \in H$

(b) When n is odd

(i) $\det({}^1A)$ and $\det({}^2A)$ both are pure imaginary

(ii) $\det(A) = -\overline{\det({}^1A)}e_1 - \overline{\det({}^2A)}e_2$

(iii) $\det(A_1)$ and $\det(A_2)$ both are pure imaginary

(iv) $\det(A) = -\overline{\det(A_1)}\ e_1 - \overline{\det(A_2)}e_2$

(v) $\det(A) \in i_1 H$ or $i_2 H$

Proof:

Let $A \in C_2^{n\times n}$ be a $i_1 i_2 -$ Skew $-$Hermitian matrix

$\Rightarrow A' = -A^t$

$\Rightarrow \det(A') = \det(-A^t)$

$\Rightarrow \det(A') = (-1)^n \det(A^t)$

$\Rightarrow \det(A') = (-1)^n \det(A) \quad (\because \det(A^t) = \det(A))$

$\Rightarrow [\det(A)]' = (-1)^n \det(A) \ (\because \det(A') = [\det(A)]')$

(a) When n is even

$[\det(A)]' = \det(A) \quad (\because (-1)^n = 1, \text{n is even})$

The results directly follow **Proposition 3.10**

(b) When n is odd

$[\det(A)]' = -\det(A) \quad (\because (-1)^n = -1, \text{n is odd})$

(i) $[\det(A)]' = -\det(A)$

$\Rightarrow [\det({}^1A)e_1 + \det({}^2A)\ e_2]'$

$= -\det({}^1A)e_1 - \det({}^2A)e_2$

$\Rightarrow \overline{\det({}^1A)}e_1 + \overline{\det({}^2A)}e_2$

$= -\det({}^1A)e_1 - \det({}^2A)e_2$

$\Rightarrow \overline{\det({}^1A)} = -\det({}^1A)$ and $\overline{\det({}^2A)} = -\det({}^2A)$

$\Rightarrow \det({}^1A)$ is Pure imaginary and $\det({}^2A)$ is Pure imaginary

(ii) From **(i)**

$\det({}^1A)$ and $\det({}^2A)$ both are Pure imaginary

$\Rightarrow \overline{\det({}^1A)} = -\det({}^1A)$ and $\overline{\det({}^2A)} = -\det({}^2A)$

$\Rightarrow \det(A) = -\overline{\det({}^1A)}e_1 - \overline{\det({}^2A)}e_2$

(iii) $[\det(A)]' = -\det(A)$

$\Rightarrow [\det(A_1)e_1 + \det(A_2)e_2]'$

$= -\det(A_1)e_1 - \det(A_2)e_2$

$\Rightarrow \overline{\det(A_1)}e_1 + \overline{\det(A_2)}e_2$

$= -\det(A_1)e_1 - \det(A_2)e_2$

$\Rightarrow \overline{\det(A_1)} = -\det(A_1)$ and $\overline{\det(A_2)} = -\det(A_2)$

$\Rightarrow \det(A_1)$ is pure imaginary and $\det(A_2)$ is pure imaginary

(iv) From **(iii)**

$\det(A_1)$ and $\det(A_2)$ both are pure imaginary

$\Rightarrow \overline{\det(A_1)} = -\det(A_1)$ and $\overline{\det(A_2)} = -\det(A_2)$

$\Rightarrow \det(A) = -\overline{\det(A_1)}\ e_1 - \overline{\det(A_2)}e_2$

(v) From **(i)**

$\det({}^1A)$ and $\det({}^2A)$ both are pure imaginary

$\Rightarrow \det(A) \in i_1 H$ or $i_2 H$

Proposition 3.14: Let $A \in C_2^{n\times n}$ be an orthogonal matrix. Then $\det(A) \in \{1, -1, i_1 i_2, -i_1 i_2\}$.

Proof: Let $A \in C_2^{n\times n}$ be an orthogonal matrix

$\Rightarrow A^t A = AA^t = I$

$\Rightarrow \det(A^t A) = \det(I)$

$\Rightarrow \det(A^t)\det(A) = 1$

$\Rightarrow \det(A)\det(A) = 1 \quad (\because \det(A^t) = \det(A))$

$\Rightarrow \{\det(A)\}^2 = 1$

$\Rightarrow \{\det({}^1A)\ e_1 + \det({}^2A)\ e_2\}^2 = 1e_1 + 1e_2$

$\Rightarrow \{\det({}^1A)\}^2 e_1 + \{\det({}^2A)\}^2 e_2 = 1e_1 + 1e_2$

$\Rightarrow \{\det({}^1A)\}^2 = 1$ and $\{\det({}^2A)\}^2 = 1$

$\Rightarrow \det({}^1A) = 1, -1$ and $\det({}^2A) = 1, -1$

Hence $\det(A) \in \{1, -1, i_1 i_2, -i_1 i_2\}$.

Note 3.9. Every Bicomplex orthogonal Matrix A is invertible and $A^{-1} = A^t$.

Proposition 3.15: Let $A \in C_2^{n\times n}$ be a i_1 − Unitary Matrix. Then

(i) $\overline{\det(^2A)}\det(^1A) = 1$ or equivalently $\overline{\det(^1A)}\det(^2A) = 1$

(ii) $\det(A) = \frac{1}{\det(^2A)}e_1 + \frac{1}{\det(^1A)}e_2$

(iii) $\det(A_1)\det(A_2) = 1$

(iv) $\det(A) = \frac{1}{\det(A_2)}e_1 + \frac{1}{\det(A_1)}e_2$

Proof:

(i) Let $A \in C_2^{n\times n}$ be a i_1 − Unitary Matrix

$\Rightarrow A^{*t}A = AA^{*t} = I$

$\Rightarrow \det(A^{*t}A) = \det(I)$

$\Rightarrow \det(A^{*t})\det(A) = 1$

$\Rightarrow \det(A^{*})\det(A) = 1 \quad (\because \det(A^{*t}) = \det(A^{*}))$

$\Rightarrow [\det(A)]^{*}\det(A) = 1 \quad (\because \det(A^{*}) = [\det(A)]^{*})$

$\Rightarrow [\det(^1A)e_1 + \det(^2A)e_2]^{*}[\det(^1A)e_1 + \det(^2A)e_2] = 1$

$\Rightarrow [\overline{\det(^1A)}e_2 + \overline{\det(^2A)}e_1][\det(^1A)e_1 + \det(^2A)e_2] = 1$

$\Rightarrow [\overline{\det(^2A)}e_1 + \overline{\det(^1A)}e_2][\det(^1A)e_1 + \det(^2A)e_2] = 1$

$\Rightarrow \overline{\det(^2A)}\det(^1A)e_1 + \overline{\det(^1A)}\det(^2A)e_2 = 1e_1 + 1e_2$

$\Rightarrow \overline{\det(^2A)}\det(^1A) = 1$ and $\overline{\det(^1A)}\det(^2A) = 1$

$\Rightarrow \overline{\det(^2A)}\det(^1A) = 1$ or equivalently $\overline{\det(^1A)}\det(^2A) = 1$

(ii) From **(i)**

$\overline{\det(^2A)}\det(^1A) = 1$ and $\overline{\det(^1A)}\det(^2A) = 1$

$\Rightarrow \det(^1A) = \frac{1}{\overline{\det(^2A)}}$ and $\det(^2A) = \frac{1}{\overline{\det(^1A)}}$

$\Rightarrow \det(^1A)e_1 + \det(^2A)e_2 = \frac{1}{\overline{\det(^2A)}}e_1 + \frac{1}{\overline{\det(^1A)}}e_2$

$\Rightarrow \det(A) = \frac{1}{\overline{\det(^2A)}}e_1 + \frac{1}{\overline{\det(^1A)}}e_2$

(iii) Let $A \in C_2^{n\times n}$ be a i_1 − Unitary Matrix

$\Rightarrow A^{*t}A = AA^{*t} = I$

$\Rightarrow \det(A^{*t}A) = \det(I)$

$\Rightarrow \det(A^{*t})\det(A) = 1$

$\Rightarrow \det(A^{*})\det(A) = 1 \quad (\because \det(A^{*t}) = \det(A^{*}))$

$\Rightarrow [\det(A)]^{*}\det(A) = 1 \quad (\because \det(A^{*}) = [\det(A)]^{*})$

$\Rightarrow [\det(A_1)e_1 + \det(A_2)e_2]^{*}[\det(A_1)e_1 + \det(A_2)e_2] = 1$

$\Rightarrow [\det(A_1)e_2 + \det(A_2)e_1][\det(A_1)e_1 + \det(A_2)e_2] = 1$

$\Rightarrow [\det(A_2)e_1 + \det(A_1)e_2][\det(A_1)e_1 + \det(A_2)e_2] = 1$

$\Rightarrow \det(A_2)\det(A_1)e_1 + \det(A_1)\det(A_2)e_1 = 1e_1 + 1e_2$

$\Rightarrow \det(A_2)\det(A_1) = 1$ and $\det(A_1)\det(A_2) = 1$

$\Rightarrow \det(A_1)\det(A_2) = 1$

(iv) From **(iii)**

$\det(A_1)\det(A_2) = 1$

$\Rightarrow \det(A_1) = \frac{1}{\det(A_2)}$ and $\det(A_2) = \frac{1}{\det(A_1)}$

$\Rightarrow \det(A_1)e_1 + \det(A_2)e_2 = \frac{1}{\det(A_2)}e_1 + \frac{1}{\det(A_1)}e_2$

$\Rightarrow \det(A) = \frac{1}{\det(A_2)}e_1 + \frac{1}{\det(A_1)}e_2$

Note 3.10: i_1 − Unitary Matrix is invertible and $A^{-1} = A^{*t}$.

Proposition 3.16. Let $A = {}^1A\,e_1 + {}^2A\,e_2 \in C_2^{n\times n}$ be a i_2 − Unitary Matrix. Then

(i) $\det(^1A)\det(^2A) = 1$

(ii) $\det(A) = \frac{1}{\det(^2A)}e_1 + \frac{1}{\det(^1A)}e_2$

(iii) $\overline{\det(A_2)}\det(A_1) = 1$ or equivalently $\overline{\det(A_1)}\det(A_2) = 1$

(iv) $\det(A) = \frac{1}{\overline{\det(A_2)}}e_1 + \frac{1}{\overline{\det(A_1)}}e_2$

Proof:

(i) Let $A = {}^1A\,e_1 + {}^2A\,e_2 \in C_2^{n\times n}$ be a i_2 − Unitary Matrix

$\Rightarrow A^{\#t}A = AA^{\#t} = I$

$\Rightarrow \det(A^{\#t}A) = \det(I)$

$\Rightarrow \det(A^{\#t})\det(A) = 1$

$\Rightarrow \det(A^{\#})\det(A) = 1 \quad (\because \det(A^{\#t}) = \det(A^{\#}))$

$\Rightarrow [\det(A)]^{\#}\det(A) = 1 \quad (\because \det(A^{\#}) = [\det(A)]^{\#})$

$\Rightarrow [\det(^1A)e_1 + \det(^2A)e_2]^{\#}[\det(^1A)e_1 + \det(^2A)e_2] = 1$

$\Rightarrow [\det(^1A)e_2 + \det(^2A)e_1][\det(^1A)e_1 + \det(^2A)e_2] = 1$

$\Rightarrow [\det(^2A)e_1 + \det(^1A)e_2][\det(^1A)e_1 + \det(^2A)e_2] = 1$

$\Rightarrow \det(^2A)\det(^1A)e_1 + \det(^1A)\det(^2A)e_2 = 1e_1 + 1e_2$

$\Rightarrow \det(^2A)\det(^1A) = 1$ and $\det(^1A)\det(^2A) = 1$

$\Rightarrow \det(^1A)\det(^2A) = 1$

(ii) From **(i)** $\det(^1A)\det(^2A) = 1$

$\Rightarrow \det(^1A) = \frac{1}{\det(^2A)}$ and $\det(^2A) = \frac{1}{\det(^1A)}$

$\Rightarrow \det(^1A)e_1 + \det(^2A)e_2 = \frac{1}{\det(^2A)}e_1 + \frac{1}{\det(^1A)}e_2$

$\Rightarrow \det(A) = \frac{1}{\det(^2A)}e_1 + \frac{1}{\det(^1A)}e_2$

(iii) Let $A = {}^1A\,e_1 + {}^2A\,e_2 \in C_2^{n\times n}$ be a i_2 − Unitary Matrix

$\Rightarrow A^{\#t}A = AA^{\#t} = I$

$\Rightarrow \det(A^{\#t}A) = \det(I)$

$\Rightarrow \det(A^{\#t})\det(A) = 1$

$\Rightarrow \det(A^{\#})\det(A) = 1 \quad (\because \det(A^{\#t}) = \det(A^{\#}))$

$\Rightarrow [\det(A)]^{\#}\det(A) = 1 \quad (\because \det(A^{\#}) = [\det(A)]^{\#})$

$\Rightarrow [\det(A_1)e_1 + \det(A_2)e_2]^{\#}[\det(A_1)e_1 + \det(A_2)e_2] = 1$

$\Rightarrow [\overline{\det(A_1)}\, e_2 + \overline{\det(A_2)}\, e_1][\det(A_1)e_1 + \det(A_2)e_2] = 1$

$\Rightarrow [\overline{\det(A_2)}\, e_1 + \overline{\det(A_1)}\, e_2][\det(A_1)e_1 + \det(A_2)e_2] = 1$

$\Rightarrow \overline{\det(A_2)}\det(A_1)e_1 + \overline{\det(A_1)}\det(A_2)\, e_2 = 1e_1 + 1e_2$

$\Rightarrow \overline{\det(A_2)}\det(A_1) = 1$ and $\overline{\det(A_1)}\det(A_2)\, e_2 = 1$

$\Rightarrow \overline{\det(A_2)}\det(A_1) = 1$ or equivalently

$\overline{\det(A_1)}\det(A_2) = 1$

(iv) From **(iii)**

$\overline{\det(A_2)}\det(A_1) = 1$ or equivalently

$\overline{\det(A_1)}\det(A_2) = 1$

$\Rightarrow \det(A_1) = \frac{1}{\overline{\det(A_2)}}$ and $\det(A_2) = \frac{1}{\overline{\det(A_1)}}$

$\Rightarrow \det(A_1)e_1 + \det(A_2)\, e_2 = \frac{1}{\overline{\det(A_2)}} e_1 + \frac{1}{\overline{\det(A_1)}} e_2$

$\Rightarrow \det(A) = \frac{1}{\overline{\det(A_2)}} e_1 + \frac{1}{\overline{\det(A_1)}} e_2$

Note 3.11. i_2 − Unitary Matrix is invertible and $A^{-1} = A^{\#t}$.

Proposition 3.17: Let $A \in C_2^{n\times n}$ be a $i_1 i_2$ − Unitary Matrix. Then $\|\det(A)\| = 1$.

Proof: Let $A \in C_2^{n\times n}$ be a $i_1 i_2$ − Unitary Matrix

$\Rightarrow A'^{t}A = AA'^{t} = I$

$\Rightarrow \det(A'^{t}A) = \det(I)$

$\Rightarrow \det(A'^{t})\det(A) = 1$

$\Rightarrow \det(A')\det(A) = 1 \quad (\because \det(A'^{t}) = \det(A'))$

$\Rightarrow [\det(A)]'\det(A) = 1 \quad (\because \det(A') = [\det(A)]')$

$\Rightarrow [\det({}^1A)e_1 + \det({}^2A)\, e_2]'[\det({}^1A)e_1 + \det({}^2A)\, e_2] = 1$

$\Rightarrow [\overline{\det({}^1A)}\, e_1 + \overline{\det({}^2A)}\, e_2][\det({}^1A)e_1 + \det({}^2A)\, e_2] = 1$

$\Rightarrow \overline{\det({}^1A)}\det({}^1A)e_1 + \overline{\det({}^2A)}\det({}^2A)e_2 = 1e_1 + 1e_2$

$\Rightarrow \overline{\det({}^1A)}\det({}^1A) = 1$ and $\overline{\det({}^2A)}\det({}^2A) = 1$

$\Rightarrow |\det({}^1A)|^2 = 1$ and $|\det({}^2A)|^2 = 1$

$\Rightarrow \|\det(A)\| = \left[\frac{|\det({}^2A)|^2 + |\det({}^2A)|^2}{2}\right]^{\frac{1}{2}} = 1$

Note 3.12. $i_1 i_2$ − Unitary Matrix is invertible and $A^{-1} = A'^{t}$.

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